

# Hypercyclic tuples of operators on $\mathbb{C}^n$ and $\mathbb{R}^n$

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## Abstract

A tuple  $(T_1, \dots, T_n)$  of continuous linear operators on a topological vector space  $X$  is called hypercyclic if there is  $x \in X$  such that the orbit of  $x$  under the action of the semigroup generated by  $T_1, \dots, T_n$  is dense in  $X$ . This concept was introduced by N. Feldman, who have raised 7 questions on hypercyclic tuples. We answer those 4 of them, which can be dealt with on the level of operators on finite dimensional spaces. In particular, we prove that the minimal cardinality of a hypercyclic tuple of operators on  $\mathbb{C}^n$  (respectively, on  $\mathbb{R}^n$ ) is  $n + 1$  (respectively,  $\frac{n}{2} + \frac{5+(-1)^n}{4}$ ), that there are non-diagonalizable tuples of operators on  $\mathbb{R}^2$  which possess an orbit being neither dense nor nowhere dense and construct a hypercyclic 6-tuple of operators on  $\mathbb{C}^3$  such that every operator commuting with each member of the tuple is non-cyclic.

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## 1 Introduction

Throughout the article  $\mathbb{K}$  stands for either the field  $\mathbb{C}$  of complex numbers or the field  $\mathbb{R}$  of real numbers,  $\mathbb{Q}$  is the field of rational numbers,  $\mathbb{Z}$  is the set of integers and  $\mathbb{Z}_+$  is the set of non-negative integers. Symbol  $L(X)$  stands for the space of continuous linear operators on a topological vector space  $X$ . A family  $\mathcal{F} = \{F_a : a \in A\}$  of continuous maps from a topological space  $X$  to a topological space  $Y$  is called *universal* if there is  $x \in X$  for which the orbit  $O(\mathcal{F}, x) = \{F_a x : a \in A\}$  is dense in  $Y$ . Such an  $x$  is called a *universal element* for  $\mathcal{F}$ . We use the symbol  $\mathcal{U}(\mathcal{F})$  to denote the set of universal elements for  $\mathcal{F}$ . If  $X$  is a topological vector space,  $n \in \mathbb{N}$  and  $\mathbf{T} = (T_1, \dots, T_n) \in L(X)^n$ , then we call  $\mathbf{T}$  a *commuting  $n$ -tuple* if  $T_j T_k = T_k T_j$  for  $1 \leq j, k \leq n$ . An  $n$ -tuple  $\mathbf{T} = (T_1, \dots, T_n) \in L(X)^n$  is called *hypercyclic* if it is commuting and the semigroup

$$\mathcal{F}_{\mathbf{T}} = \{T_1^{k_1} \dots T_n^{k_n} : k_j \in \mathbb{Z}_+\}$$

is universal. The elements of  $\mathcal{U}(\mathcal{F}_{\mathbf{T}})$  are called *hypercyclic vectors for  $\mathbf{T}$* . This concept was introduced and studied by Feldman [3]. In the case  $n = 1$ , it becomes the conventional hypercyclicity, which has been widely studied, see the book [1] and references therein. It turns out that, unlike for the classical hypercyclicity, there are hypercyclic tuples of operators on finite dimensional spaces. Namely, in [3] it is shown that  $\mathbb{C}^n$  admits a hypercyclic  $(n + 1)$ -tuple of operators. It is also shown that for every tuple of operators on  $\mathbb{C}^n$ , any orbit is either dense or is nowhere dense and that the latter fails for tuples of operators on  $\mathbb{R}^n$ . The article [3] culminates in raising 7 open questions. We answer those 4 of them (Questions 2, 3, 5 and 6 in the list) that can be dealt with on the level of operators on finite dimensional spaces.

**Question F1.** *If  $\mathbf{T}$  is a hypercyclic tuple, then must  $\mathcal{F}_{\mathbf{T}}$  contain a cyclic operator?*

**Question F2.** *If  $\mathbf{T}$  is a hypercyclic tuple, is there a cyclic operator that commutes with  $\mathbf{T}$ ?*

**Question F3.** *Is there a hypercyclic  $n$ -tuple of operators on  $\mathbb{C}^n$ ?*

**Question F4.** *Are there non-diagonalizable commuting tuples of operators on  $\mathbb{R}^n$ , possessing orbits that are neither dense nor nowhere dense?*

We answer the above questions by considering dense additive subsemigroups of  $\mathbb{R}^n$  and by studying cyclic commutative subalgebras of full matrix algebras.

As usual, we identify  $L(\mathbb{K}^n)$  with the  $\mathbb{K}$ -algebra of  $n \times n$  matrices with entries from  $\mathbb{K}$ . In this way  $L(\mathbb{R}^n)$  will be treated as an  $\mathbb{R}$ -subalgebra of the  $\mathbb{C}$ -algebra  $L(\mathbb{C}^n)$ . For a commutative subalgebra  $\mathbb{A}$  of  $L(\mathbb{K}^n)$ , by  $\mathbb{A}^+$  we denote the subalgebra of  $S \in L(\mathbb{K}^n)$  commuting with each element of  $\mathbb{A}$ . Clearly  $\mathbb{A}$  is a subalgebra of  $\mathbb{A}^+$ , however  $\mathbb{A}^+$  may fail to be commutative. Recall that a subalgebra  $\mathbb{A}$  of  $L(\mathbb{K}^n)$  is called *cyclic* if there is  $x \in \mathbb{K}^n$  such that  $\{Ax : A \in \mathbb{A}\} = \mathbb{K}^n$ . For a commuting tuple  $\mathbf{T}$  of operators on  $\mathbb{K}^n$  we denote the unital subalgebra of  $L(\mathbb{K}^n)$  generated by  $\mathbf{T}$  by the symbol  $\mathbb{A}_{\mathbf{T}}$ . Obviously,  $\mathbb{A}_{\mathbf{T}}$  is commutative and therefore  $\mathbb{A}_{\mathbf{T}} \subseteq \mathbb{A}_{\mathbf{T}}^+$  and  $\mathbb{A}_{\mathbf{T}}$  is cyclic if  $\mathbf{T}$  is hypercyclic. Thus each hypercyclic tuple of operators on  $\mathbb{K}^n$  lies in a unital commutative cyclic subalgebra of  $L(\mathbb{K}^n)$ .

Recall that a *character* on a complex algebra  $\mathbb{A}$  is a non-zero algebra homomorphism from  $\mathbb{A}$  to  $\mathbb{C}$ . It is well-known that the set of characters is always linearly independent in the space of linear functionals on  $\mathbb{A}$ . Moreover, the set of characters on a unital commutative Banach algebra is always non-empty. Hence an  $n$ -dimensional unital commutative complex algebra has at least 1 and at most  $n$  characters. For a finite dimensional complex algebra  $\mathbb{A}$ ,  $\varkappa(\mathbb{A})$  stands for the number of characters on  $\mathbb{A}$ . Thus

$$1 \leq \varkappa(\mathbb{A}) \leq n \text{ for every } n\text{-dimensional unital commutative complex algebra } \mathbb{A}. \quad (1.1)$$

For a real algebra  $\mathbb{A}$ , a *character* on  $\mathbb{A}$  is a non-zero  $\mathbb{R}$ -algebra homomorphism from  $\mathbb{A}$  to  $\mathbb{C}$ . It is well-known and easy to see that each character on  $\mathbb{A}$  has a unique extension to a character on the complex algebra  $\mathbb{A}_{\mathbb{C}} = \mathbb{A} \oplus i\mathbb{A} = \mathbb{C} \otimes \mathbb{A}$ , being the complexification of  $\mathbb{A}$ . Thus the characters on  $\mathbb{A}$  and the characters on  $\mathbb{A}_{\mathbb{C}}$  are in the natural one-to-one correspondence. If  $\chi$  is a character on a real algebra  $\mathbb{A}$ , then its complex conjugate  $\bar{\chi}$  is also a character on  $\mathbb{A}$ . Clearly,  $\chi(\mathbb{A})$  being a non-trivial  $\mathbb{R}$ -subalgebra of  $\mathbb{C}$  must coincide either with  $\mathbb{R}$  or with  $\mathbb{C}$ . The characters of the second type come in complex conjugate pairs. For a finite dimensional  $\mathbb{R}$ -algebra, by  $\varkappa_0(\mathbb{A})$  we denote the number of complex conjugate pairs of characters  $\chi$  on  $\mathbb{A}$  satisfying  $\chi(\mathbb{A}) = \mathbb{C}$ . Similarly,  $\varkappa_1(\mathbb{A})$  stands for the number of real-valued characters on  $\mathbb{A}$ . Since  $\varkappa(\mathbb{A}_{\mathbb{C}}) = \varkappa_1(\mathbb{A}) + 2\varkappa_0(\mathbb{A})$ , (1.1) implies that

$$1 \leq 2\varkappa_0(\mathbb{A}) + \varkappa_1(\mathbb{A}) \leq n \text{ for every unital commutative } n\text{-dimensional real algebra } \mathbb{A}. \quad (1.2)$$

**Lemma 1.1.** *Let  $\mathbb{A}$  be a commutative cyclic subalgebra of  $L(\mathbb{K}^n)$ . Then  $I \in \mathbb{A}$ ,  $\mathbb{A} = \mathbb{A}^+$  and  $\dim \mathbb{A} = n$ . In particular,  $1 \leq \varkappa(\mathbb{A}) \leq n$  if  $\mathbb{K} = \mathbb{C}$  and  $1 \leq \varkappa_1(\mathbb{A}) + 2\varkappa_0(\mathbb{A}) \leq n$  if  $\mathbb{K} = \mathbb{R}$ .*

*Proof.* Let  $x \in \mathbb{K}^n$  be a cyclic vector for  $\mathbb{A}$ . Then the linear map  $A \mapsto Ax$  from  $\mathbb{A}$  to  $\mathbb{K}^n$  is surjective and therefore  $\dim \mathbb{A} \geq n$ . Now let  $B \in \mathbb{A}^+$  be such that  $Bx = 0$ . Since  $B$  commutes with each member of  $\mathbb{A}$ ,  $\ker B$  is invariant for every element of  $\mathbb{A}$  and therefore  $\mathbb{K}^n = \{Ax : A \in \mathbb{A}\} \subseteq \ker B$ . That is,  $B = 0$  whenever  $B \in \mathbb{A}^+$  and  $Bx = 0$ . Thus the linear map  $B \mapsto Bx$  from  $\mathbb{A}^+$  to  $\mathbb{K}^n$  is injective and therefore  $\dim \mathbb{A}^+ \leq n$ . Since also  $\mathbb{A} \subseteq \mathbb{A}^+$ , we get  $\mathbb{A} = \mathbb{A}^+$  and  $\dim \mathbb{A} = n$ . Since  $I \in \mathbb{A}^+$ , we have  $I \in \mathbb{A}$ . The required estimates now follow from (1.1) and (1.2).  $\square$

The significance of Lemma 1.1 becomes clear in view of the next two results.

**Theorem 1.2.** *Let  $\mathbb{A}$  be a commutative cyclic subalgebra of  $L(\mathbb{C}^n)$  and  $m = 2n - \varkappa(\mathbb{A}) + 1$ . Then  $\mathbb{A}$  contains a hypercyclic  $m$ -tuple of operators on  $\mathbb{C}^n$  and  $\mathbb{A}$  contains no hypercyclic  $(m - 1)$ -tuples.*

**Theorem 1.3.** *Let  $\mathbb{A}$  be a commutative cyclic subalgebra of  $L(\mathbb{R}^n)$  and  $r = n - \varkappa_0(\mathbb{A}) + 1$ . Then  $\mathbb{A}$  contains a hypercyclic  $r$ -tuple of operators on  $\mathbb{R}^n$ , while every orbit of every  $(r - 1)$ -tuple of operators from  $\mathbb{A}$  is nowhere dense.*

The subalgebra  $\mathbb{A}_{\text{diag}}$  of  $L(\mathbb{C}^n)$  consisting of all operators with diagonal matrices is commutative, cyclic and has exactly  $n$  characters. It is also easy to show that a commutative subalgebra  $\mathbb{A}$  of  $L(\mathbb{C}^n)$  has  $n$  characters if and only if  $\mathbb{A}$  is conjugate to  $\mathbb{A}_{\text{diag}}$ . Now if  $\mathbb{A}_0$  is the subalgebra of  $L(\mathbb{C}^2)$  consisting of the matrices of the form  $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$  with  $a, b \in \mathbb{C}$ , then  $\mathbb{A}_0$  is commutative and cyclic

and  $\varkappa(\mathbb{A}_0) = 1$ . The direct sum of  $\mathbb{A}_0$  and  $n - 2$  copies of  $L(\mathbb{C}^1)$  provides a commutative cyclic subalgebra of  $L(\mathbb{C}^n)$  with exactly  $n - 1$  characters. By Theorem 1.2, it contains a hypercyclic  $(n + 2)$ -tuple. Thus Theorem 1.2 implies the following corollary, which answers Question F3 negatively.

**Corollary 1.4.** *The minimal cardinality of a hypercyclic tuple of operators on  $\mathbb{C}^n$  is exactly  $n + 1$ . The minimal cardinality of a non-diagonalizable hypercyclic tuple of operators on  $\mathbb{C}^n$  is  $n + 2$ .*

If we consider the subalgebra  $\mathbb{A}_1$  of  $L(\mathbb{R}^2)$  consisting of matrices of the form  $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$  with  $a, b \in \mathbb{R}$ , then  $\mathbb{A}_1$  is commutative and cyclic and  $\varkappa_0(\mathbb{A}_1) = 1$ . Then the direct sum  $\mathbb{A}_m$  of  $m$  copies of  $\mathbb{A}_1$  is a cyclic commutative subalgebra of  $L(\mathbb{R}^{2m})$  with  $\varkappa_0(\mathbb{A}_m) = m$ . Similarly, the direct sum  $\mathbb{A}'_m$  of  $L(\mathbb{R}^1)$  and  $m$  copies of  $\mathbb{A}_1$  is a cyclic commutative subalgebra of  $L(\mathbb{R}^{2m+1})$  with  $\varkappa_0(\mathbb{A}'_m) = m$ . Thus for every positive integer  $n$ , there is a commutative cyclic subalgebra  $\mathbb{A}$  of  $L(\mathbb{R}^n)$  with  $\varkappa_0(\mathbb{A})$  being exactly the integer part of  $\frac{n}{2}$ . This observation together with Theorem 1.3 implies the following corollary.

**Corollary 1.5.** *Let  $n$  be a positive integer and  $k = \frac{n}{2} + \frac{5+(-1)^{n+1}}{4}$ . Then there is a hypercyclic  $k$ -tuple of operators on  $\mathbb{R}^n$ . Moreover, every orbit of every commuting  $(k - 1)$ -tuple of operators on  $\mathbb{R}^n$  is nowhere dense.*

Note that  $\frac{n}{2} + \frac{5+(-1)^{n+1}}{4}$  is  $\frac{n}{2} + 1$  if  $n$  is even and is  $\frac{n+3}{2}$  if  $n$  is odd. The following result provides negative answers to Questions F1 and F2.

**Proposition 1.6.** *There is a hypercyclic 6-tuple  $\mathbf{T}$  of operators on  $\mathbb{C}^3$  such that every operator commuting with (all members of)  $\mathbf{T}$  is non-cyclic. Moreover, there is a hypercyclic 4-tuple  $\mathbf{T}$  of operators on  $\mathbb{R}^3$  such that every operator commuting with  $\mathbf{T}$  is non-cyclic.*

Question F4 admits an affirmative answer by means of the following proposition.

**Proposition 1.7.** *There exist positive numbers  $a_1, a_2, a_3$  and non-zero real numbers  $b_1, b_2, b_3$  such that the commuting triple  $\mathbf{T} = (T_1, T_2, T_3)$  of non-diagonalizable (even when considered as members of  $L(\mathbb{C}^2)$ ) operators  $T_j = \begin{pmatrix} a_j & b_j \\ 0 & a_j \end{pmatrix}$  on  $\mathbb{R}^2$  has the following properties:*

- (a1)  $\mathbf{T}$  is non-hypercyclic;
- (a2) the orbit of  $x = (x_1, x_2) \in \mathbb{R}^2$  with respect to  $\mathbf{T}$  is contained and is dense in the half-plane  $\Pi = \{(s, t) \in \mathbb{R}^2 : t > 0\}$  if  $x_2 > 0$ .

## 2 Dense additive subsemigroups of $\mathbb{R}^n$

We start with the following trivial lemma, whose proof we give for sake of completeness.

**Lemma 2.1.** *Let  $H$  be an additive subgroup of  $\mathbb{R}^n$  with at most  $n$  generators. Then  $H$  is nowhere dense in  $\mathbb{R}^n$ .*

*Proof.* Let  $k \leq n$  and  $x_1, \dots, x_k$  be generators of  $H$ . If the linear span  $L$  of  $x_1, \dots, x_k$  differs from  $\mathbb{R}^n$ , then  $H$  is nowhere dense as a subset of the closed nowhere dense set  $L$ . It remains to consider the case  $L = \mathbb{R}^n$ . Since  $k \leq n$ , it follows that  $k = n$  and the vectors  $x_1, \dots, x_k$  form a basis in  $\mathbb{R}^n$ . Hence  $H$  is a discrete lattice in  $\mathbb{R}^n$  and therefore is nowhere dense in  $\mathbb{R}^n$ .  $\square$

**Lemma 2.2.** *Let  $\alpha_1, \dots, \alpha_n$  be  $n$  positive numbers linearly independent over  $\mathbb{Q}$ . Then*

$$\Omega = \{(m_1 - m_0\alpha_1, \dots, m_n - m_0\alpha_n) : m_0, \dots, m_n \in \mathbb{Z}_+\} \quad \text{is dense in } \mathbb{R}^n.$$

*Proof.* Let  $g = \alpha + \mathbb{Z}^n \in \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ . Since the components of  $\alpha = (\alpha_1, \dots, \alpha_n)$  are linearly independent over  $\mathbb{Q}$ , the classical Kronecker theorem implies that  $\{mg : m \in \mathbb{Z}_+\}$  is dense in the compact metrizable topological group  $\mathbb{T}^n$ . It follows that for every  $x \in \mathbb{R}^n$ , we can find a strictly increasing sequence  $\{m_{0,j}\}_{j \in \mathbb{Z}_+}$  in  $\mathbb{Z}_+$  such that  $m_{0,j}g \rightarrow -x + \mathbb{Z}^n$ . Hence we can pick sequences  $\{m_{l,j}\}_{j \in \mathbb{Z}_+}$  of integers for  $1 \leq l \leq n$  such that  $m_{l,j} - m_{0,j}\alpha_l \rightarrow x_l$  as  $j \rightarrow \infty$  for  $1 \leq l \leq n$ . Since  $m_{0,j} \rightarrow +\infty$  and  $\alpha_l > 0$ , we have  $m_{l,j} \geq 0$  for all sufficiently large  $j$ . It follows that  $x$  is an accumulation point of  $\Omega$ . Since  $x \in \mathbb{R}^n$  is arbitrary,  $\Omega$  is dense in  $\mathbb{R}^n$ .  $\square$

**Lemma 2.3.** *Let  $x_1, \dots, x_n$  be a basis in  $\mathbb{R}^n$  and  $G$  be a finite abelian group (carrying the discrete topology) with an  $n+1$ -element generating set  $\{g_0, \dots, g_n\}$ . Then there is  $x_0 \in \mathbb{R}^n$  such that the subsemigroup of  $G \times \mathbb{R}^n$  generated by  $(g_0, x_0), \dots, (g_n, x_n)$  is dense in  $G \times \mathbb{R}^n$ .*

*Proof.* For the sake of homogeneity, we use the additive notation for the operation on  $G$ . Let  $\alpha_1, \dots, \alpha_n$  be positive numbers linearly independent over  $\mathbb{Q}$  and  $x_0 = -(\alpha_1 x_1 + \dots + \alpha_n x_n)$ . It suffices to show that the subsemigroup  $H$  of  $G \times \mathbb{R}^n$  generated by  $(g_0, x_0), \dots, (g_n, x_n)$  is dense in  $G \times \mathbb{R}^n$ . Let  $m$  be the order of  $G$  and  $h \in G$ . Pick positive integers  $j_0, \dots, j_n$  such that  $h = j_0 g_0 + \dots + j_n g_n$ . Then

$$H_h \subset H, \text{ where } H_h = \{(j_0 + k_0 m)(g_0, x_0) + \dots + (j_n + k_n m)(g_n, x_n) : k_j \in \mathbb{Z}_+\}.$$

Denote  $y = j_0 x_0 + \dots + j_n x_n$ . Using the equalities  $h = j_0 g_0 + \dots + j_n g_n$ ,  $mg_l = 0$  for  $0 \leq l \leq n$  and the above display, we obtain

$$H_h = \{(h, y + m((k_1 - \alpha_1 k_0)x_1 + \dots + (k_n - \alpha_n k_0)x_n)) : k_j \in \mathbb{Z}_+\}.$$

By Lemma 2.2,  $H_h$  is dense in  $\{h\} \times \mathbb{R}^n$ . Since  $h$  is an arbitrary element of  $G$  and  $H_h \subset H$ ,  $H$  is dense in  $G \times \mathbb{R}^n$ .  $\square$

Applying Lemma 2.3 in the case  $G = \{0\}$  and  $g_0 = \dots = g_n = 0$ , we get the following corollary.

**Corollary 2.4.** *Let  $x_1, \dots, x_n$  be a basis in  $\mathbb{R}^n$ . Then there exists  $x_0 \in \mathbb{R}^n$  such that the additive subsemigroup  $H$  in  $\mathbb{R}^n$  generated by  $\{x_0, \dots, x_n\}$  is dense in  $\mathbb{R}^n$ .*

### 3 Properties of the exponential map

Throughout this section

$\mathbb{A}$  is a finite dimensional unital commutative algebra over  $\mathbb{K}$

$$\text{and } e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!} \in \mathbb{A} \text{ for } a \in \mathbb{A}.$$

The symbol  $\exp_{\mathbb{A}}$  stands for the map  $a \mapsto e^a$  from  $\mathbb{A}$  to itself. Since  $\mathbb{A}$  is commutative,  $e^{a+b} = e^a e^b$  and  $e^a$  is invertible with the inverse  $e^{-a}$  for any  $a, b \in \mathbb{A}$ . Thus  $\exp_{\mathbb{A}}$  is a homomorphism from the additive group  $(\mathbb{A}, +)$  to the (abelian) group  $(\mathbb{A}^*, \cdot)$  of invertible elements of  $\mathbb{A}$ . Since  $\exp_{\mathbb{A}}$  is a smooth map with the Jacobian having rank  $\dim \mathbb{A}$  at every point,  $\exp_{\mathbb{A}}$  is a local diffeomorphism and therefore a local homeomorphism from  $\mathbb{A}$  to  $\mathbb{A}$ . The following lemma represents a well-known fact. We sketch its proof for the sake of convenience.

**Lemma 3.1.** *Let  $\mathbb{K} = \mathbb{C}$  and  $\Omega$  be the set of all characters on  $\mathbb{A}$ . Then there are  $p_{\chi} \in \mathbb{A}$  for  $\chi \in \Omega$  such that*

- (a)  $p_{\chi} p_{\varphi} = \delta_{\chi, \varphi} p_{\chi}$  for every  $\chi, \varphi \in \Omega$  and  $\sum_{\chi \in \Omega} p_{\chi} = \mathbf{1}$ ;
- (b) the spectrum of  $ap_{\chi}$  in the subalgebra  $\mathbb{A}_{\chi} = p_{\chi} \mathbb{A}$  is  $\{\chi(a)\}$  for every  $a \in \mathbb{A}$  and  $\chi \in \Omega$ .

Moreover, conditions (a) and (b) determine the idempotents  $p_\chi$  uniquely.

*Sketch of the proof.* By (1.1),  $\Omega$  is a finite set. Since  $\Omega$  is linearly independent, we can pick  $a \in \mathbb{A}$  such that the numbers  $\chi(a)$  for  $\chi \in \Omega$  are pairwise different. For  $\chi \in \Omega$  we take a circle  $\Gamma_\chi$  on the complex plane centered at  $\chi(a)$  and such that  $\varphi(a)$  for  $\varphi \neq \chi$  are all outside the closed disk encircled by  $\Gamma_\chi$ . Since  $\sigma(a) = \{\varphi(a) : \varphi \in \Omega\}$ ,  $\Gamma_\chi \cap \sigma(a) = \emptyset$  and we can consider

$$p_\chi = \frac{1}{2\pi i} \oint_{\Gamma_\chi} (a - \zeta \mathbf{1})^{-1} d\zeta \in \mathbb{A},$$

where  $\Gamma_\chi$  is encircled counterclockwise. In exactly the same way as when dealing with the Riesz projections, it is straightforward to see that  $p_\chi$  are pairwise orthogonal idempotents and form a partition of the identity. Thus (a) is satisfied. It is also a routine exercise to check that  $b - \sum_{\chi \in \Omega} \chi(b) p_\chi$  is nilpotent for every  $b \in \mathbb{A}$ , which implies (b). The uniqueness is also standard: one has to check that any  $p_\chi$  satisfying (a) and (b) must actually satisfy the above display as well.  $\square$

**Lemma 3.2.** *Let  $\mathbb{K} = \mathbb{C}$  and  $k = \kappa(\mathbb{A})$ . Then the homomorphism  $\exp_{\mathbb{A}} : \mathbb{A} \rightarrow \mathbb{A}^*$  is a surjective local homeomorphism, whose kernel  $\ker \exp_{\mathbb{A}}$  is a subgroup of  $(\mathbb{A}, +)$  generated by  $k$  linearly independent elements.*

*Proof.* We already know that  $\exp_{\mathbb{A}}$  is a local homeomorphism. Let us verify that  $\exp_{\mathbb{A}} : \mathbb{A} \rightarrow \mathbb{A}^*$  is surjective. Indeed, let  $a \in \mathbb{A}^*$ . Since  $0 \notin \sigma(a)$ , we can pick  $w \in \mathbb{C} \setminus \{0\}$  such that the ray  $L = \{tw : t \geq 0\}$  does not meet  $\sigma(a)$ . Now let  $\varphi : \mathbb{C} \setminus L \rightarrow \mathbb{C}$  be a branch of the logarithm function. Since  $\varphi$  is holomorphic on the open set  $\mathbb{C} \setminus L$  containing the spectrum of  $a$  we can define  $b = \varphi(a)$  in the standard holomorphic functional calculus sense. Since  $e^{\varphi(z)} = z$  for every  $z \in \mathbb{C} \setminus L$ , it follows that  $e^b = a$ . Hence  $\exp_{\mathbb{A}} : \mathbb{A} \rightarrow \mathbb{A}^*$  is surjective.

Now let  $\Omega$  be the set of characters on  $\mathbb{A}$  and  $p_\chi$  for  $\chi \in \Omega$  be the idempotents provided by Lemma 3.1. It is well-known and easy to see that the exponential of a linear map  $S$  on  $\mathbb{C}^n$  is the identity if and only if  $S$  is diagonalizable and  $\sigma(S) \subseteq 2\pi i\mathbb{Z}$ . Thus  $a \in \mathbb{A}$  belongs to  $K = \ker \exp_{\mathbb{A}}$  if and only if the linear map  $x \mapsto ax$  on  $\mathbb{A}$  is diagonalizable and its spectrum is contained in  $2\pi i\mathbb{Z}$ . From the conditions (a) and (b) of Lemma 3.1 it follows that  $a \in K$  if and only if  $a$  is a linear combination of  $p_\chi$  with coefficients from  $2\pi i\mathbb{Z}$ . Thus the  $k$  linearly independent elements  $2\pi i p_\chi$  for  $\chi \in \Omega$  generate  $K$  as a subgroup of  $(\mathbb{A}, +)$ .  $\square$

The case  $\mathbb{K} = \mathbb{R}$  turns out to be slightly more sophisticated. We start with the following curious elementary lemma.

**Lemma 3.3.** *Let  $\mathbb{K} = \mathbb{R}$ . Then for  $a \in \mathbb{A}$  the following statements are equivalent:*

- (i) *there is  $b \in \mathbb{A}$  such that  $a = e^b$ ;*
- (ii)  *$a$  is invertible and there is  $c \in \mathbb{A}$  such that  $a = c^2$ .*

*Proof.* If  $a = e^b$ , then  $a$  is invertible and  $a = (e^{b/2})^2$ . Thus (i) implies (ii). Assume now that  $a$  is invertible and  $a = c^2$  for some  $c \in \mathbb{A}$ . Then  $c$  is also invertible in  $\mathbb{A}$  and therefore  $c$  is invertible in the complexification  $\mathbb{A}_{\mathbb{C}} = \mathbb{A} \oplus i\mathbb{A}$ . By Lemma 3.2, there is  $d \in \mathbb{A}_{\mathbb{C}}$  such that  $e^d = c$ . Consider the involution  $(x + iy)^\dagger = x - iy$  on  $\mathbb{A}_{\mathbb{C}}$ , where  $x, y \in \mathbb{A}$ . Then  $e^{d^\dagger} = (e^d)^\dagger = c^\dagger = c$ . Thus  $e^{d+d^\dagger} = e^d e^{d^\dagger} = c^2 = a$ . It remains to notice that  $d + d^\dagger \in \mathbb{A}$  to conclude that (ii) implies (i).  $\square$

**Lemma 3.4.** *Let  $\mathbb{K} = \mathbb{R}$ ,  $k = \kappa_0(\mathbb{A})$  and  $m = \kappa_1(\mathbb{A})$ . Then  $\exp_{\mathbb{A}} : \mathbb{A} \rightarrow \mathbb{A}^*$  is a local homeomorphism and there is a finite subgroup  $G$  of  $\mathbb{A}^*$  isomorphic to  $\mathbb{Z}_2^m$  such that  $\mathbb{A}^*$  is the topological internal direct product of its subgroups  $\exp_{\mathbb{A}}(\mathbb{A})$  and  $G$ . Furthermore,  $\ker \exp_{\mathbb{A}}$  is a discrete subgroup of  $(\mathbb{A}, +)$  generated by  $k$  linearly independent elements.*

*Proof.* Since  $\varkappa(\mathbb{A}_{\mathbb{C}}) = 2\varkappa_0(\mathbb{A}) + \varkappa_1(\mathbb{A}) = 2k + m$ , we can enumerate the  $(2k + m)$ -element set  $\Omega$  of characters on  $\mathbb{A}_{\mathbb{C}}$  in the following way:  $\Omega = \{\chi_1, \dots, \chi_{2k+m}\}$ , where  $\chi_{2k+j}(\mathbb{A}) = \mathbb{R}$  for  $1 \leq j \leq m$  and  $\chi_{m+j}(a) = \overline{\chi_j(a)}$  for  $1 \leq j \leq k$  for any  $a \in \mathbb{A}$ . Consider the involution  $a \mapsto a^\dagger$  on  $\mathbb{A}_{\mathbb{C}} = \mathbb{A} \oplus i\mathbb{A}$  defined by the formula  $(b + ic)^\dagger = b - ic$  for  $b, c \in \mathbb{A}$ . The above properties of  $\chi_j$  can be rewritten in the following way:

$$\chi_{2k+j}(a^\dagger) = \overline{\chi_{2k+j}(a)} \quad \text{and} \quad \chi_{k+l}(a^\dagger) = \overline{\chi_l(a)} \quad \text{for } a \in \mathbb{A}_{\mathbb{C}}, 1 \leq j \leq m \text{ and } 1 \leq l \leq k. \quad (3.1)$$

Let  $p_j = p_{\chi_j} \in \mathbb{A}_{\mathbb{C}}$  for  $1 \leq j \leq 2k + m$  be the idempotents provided by Lemma 3.1 applied to the complex algebra  $\mathbb{A}_{\mathbb{C}}$ . By (3.1), the idempotents  $p_{k+1}^\dagger, \dots, p_{2k}^\dagger, p_1^\dagger, \dots, p_k^\dagger, p_{2k+1}^\dagger, \dots, p_{2k+m}^\dagger$  satisfy the conditions (a) and (b) of Lemma 3.1 in relation to the characters  $\chi_1, \dots, \chi_{2k+m}$ . By the uniqueness part of Lemma 3.1,  $p_j = p_j^\dagger$  for  $2k + 1 \leq j \leq 2k + m$  and  $p_j^\dagger = p_{k+j}$  for  $1 \leq j \leq k$ . It immediately follows that

$$p_j \in \mathbb{A} \text{ for } 2k + 1 \leq j \leq 2k + m \text{ and } zp_j + \bar{z}p_{k+j} \in \mathbb{A} \text{ for } 1 \leq j \leq k \text{ and any } z \in \mathbb{C}. \quad (3.2)$$

As we have already observed in the proof of Lemma 3.2, the kernel  $K_0$  of the homomorphism  $a \mapsto e^a$  from  $\mathbb{A}_{\mathbb{C}}$  to  $\mathbb{A}_{\mathbb{C}}^*$  is given by

$$K_0 = \{2\pi i(n_1 p_1 + \dots + n_{2k+m} p_{2k+m}) : n_j \in \mathbb{Z}\}.$$

Clearly the kernel  $K$  of  $\exp_{\mathbb{A}}$  satisfies  $K = K_0 \cap \mathbb{A}$ . Let  $b = 2\pi i(n_1 p_1 + \dots + n_{2k+m} p_{2k+m})$  with  $n_j \in \mathbb{Z}$  be an arbitrary element of  $K_0$ . By Lemma 3.1,  $\chi_j(p_l) = \delta_{j,l}$  for  $1 \leq j, l \leq 2k + m$ . Hence  $\chi_j(b) = 2\pi i n_j$  for  $1 \leq j \leq 2k + m$ . Since  $b \in \mathbb{A}$  if and only if  $b = b^\dagger$ , equalities (3.1) imply that  $b \notin \mathbb{A}$  unless  $n_j = 0$  for  $2k + 1 \leq j \leq 2k + m$  and  $n_{k+j} = -n_j$  for  $1 \leq j \leq k$ . In the latter case  $b \in \mathbb{A}$  since it is a linear combination with integer coefficients of  $2\pi i(p_j - p_{k+j})$  for  $1 \leq j \leq k$ , which all belong to  $\mathbb{A}$  according to (3.2). Summarizing, we see that  $K = K_0 \cap \mathbb{A}$  is a subgroup of  $(\mathbb{A}, +)$  generated by  $k$  linearly independent vectors  $2\pi i(p_j - p_{k+j})$  for  $1 \leq j \leq k$ .

Next, according to (3.2), for every  $\varepsilon \in \{-1, 1\}^{k+m}$ ,

$$b_\varepsilon = \sum_{j=1}^m \varepsilon_j p_{2k+j} + \sum_{j=1}^k \varepsilon_{m+j} (p_j + p_{k+j}) \in \mathbb{A}.$$

Moreover, since  $p_l$  are pairwise orthogonal idempotents forming a partition of the identity, we easily see that  $b_\varepsilon = \mathbf{1}$  if  $\varepsilon_1 = \dots = \varepsilon_{k+m} = 1$  and  $b_\varepsilon b_\delta = b_\alpha$  for every  $\varepsilon, \delta \in \{-1, 1\}^{k+m}$ , where  $\alpha_j = \varepsilon_j \delta_j$  for  $1 \leq j \leq k + m$ . It immediately follows that  $G_0 = \{b_\varepsilon : \varepsilon \in \{-1, 1\}^{k+m}\}$  is a subgroup of  $\mathbb{A}^*$  isomorphic to  $\mathbb{Z}_2^{k+m}$ . In particular,  $b_\varepsilon^2 = \mathbf{1}$  for each  $\varepsilon$ . Hence  $G$  is contained in the kernel of the homomorphism  $S : \mathbb{A}^* \rightarrow \mathbb{A}^*$ ,  $Sa = a^2$ . First, we show that  $G_0 = \ker S$ . Indeed, let  $a \in \ker S$ . Then the square of the linear map  $\hat{a}(x) = ax$  on  $\mathbb{A}_{\mathbb{C}}$  is the identity. It follows that  $\hat{a}$  is diagonalizable and its spectrum is contained in  $\{-1, 1\}$ . According to Lemma 3.1,  $a = \sum_{j=1}^{2k+m} \alpha_j p_j$  with  $\alpha_j \in \{-1, 1\}$ .

The relations  $p_j = p_j^\dagger$  for  $2k + 1 \leq j \leq 2k + m$  and  $p_j^\dagger = p_{k+j}$  for  $1 \leq j \leq k$  and linear independence of  $p_j$  imply now that  $a$  belongs to  $\mathbb{A}$  if and only if  $\alpha_j = \alpha_{k+j}$  for  $1 \leq j \leq k$ . The latter means that  $a$  coincides with one of  $b_\varepsilon$ . Thus  $G_0 = \ker S$ . Now let

$$M = \{a \in \mathbb{A} : \chi_j(a) > 0 \text{ for } 2k + 1 \leq j \leq 2k + m\}$$

and

$$G = \{b_\varepsilon : \varepsilon_{m+1} = \dots = \varepsilon_{m+k} = 1\}.$$

It is straightforward to see that  $M$  is a subgroup of  $\mathbb{A}^*$ , that  $M$  is a closed and open subset of  $\mathbb{A}^*$  and that  $G$  is a subgroup of  $G_0$  isomorphic to  $\mathbb{Z}_2^m$ . Next observe that  $\mathbb{A}^*$  is the algebraic and topological internal direct product of its subgroups  $G$  and  $M$ . Indeed, the equality  $M \cap G = \{\mathbf{1}\}$  is obvious. For

$a \in \mathbb{A}^*$ , we put  $\varepsilon_j = 1$  for  $m+1 \leq j \leq m+k$ ,  $\varepsilon_j = 1$  if  $1 \leq j \leq m$  and  $\chi_{2k+j}(a) > 0$  and  $\varepsilon_j = -1$  if  $1 \leq j \leq m$  and  $\chi_{2k+j}(a) < 0$ . Then  $b_\varepsilon \in G$  and  $\chi_j(ab_\varepsilon) = \chi_j(a)\chi_j(b_\varepsilon) > 0$  for  $2k+1 \leq j \leq 2k+m$  and therefore  $ab_\varepsilon \in M$ . It follows that  $a$  is the product of  $ab_\varepsilon \in M$  and  $b_\varepsilon \in G$ . Thus  $\mathbb{A}^*$  is the algebraic internal direct product of  $G$  and  $M$ . The fact that this product is also topological with  $G$  carrying the discrete topology easily follows from finiteness of  $G$  and the fact that  $M$  is closed and open in  $\mathbb{A}^*$ . In order to complete the proof it remains to verify that  $M = \exp_{\mathbb{A}}(\mathbb{A})$ . By Lemma 3.3, it suffices to show that  $M = S(\mathbb{A}^*)$ . If  $a \in S(\mathbb{A}^*)$ , then  $a = b^2$  with  $b \in \mathbb{A}^*$ . Invertibility of  $b$  implies that  $\chi_j(b) \in \mathbb{R} \setminus \{0\}$  for  $2k+1 \leq j \leq 2k+m$ . Hence  $\chi_j(a) = \chi_j(b^2) = \chi_j(b)^2 > 0$  for  $2k+1 \leq j \leq 2k+m$ . Thus  $a \in M$  and therefore  $S(\mathbb{A}^*) \subseteq M$ . Now let  $\mathbb{C}_- = \mathbb{C} \setminus (-\infty, 0]$  and

$$M_0 = \{a \in M : \chi_j(a) \in \mathbb{C}_- \text{ for } 1 \leq j \leq k\}.$$

Clearly  $M_0$  is an open subset of  $M$ . Moreover,  $M_0$  is dense in  $M$ . Indeed, it is easy to see that the map  $\Phi = (\chi_1, \dots, \chi_k) : \mathbb{A} \rightarrow \mathbb{C}^k$  is surjective (otherwise the characters  $\chi_1, \dots, \chi_{2k}$  on  $\mathbb{A}_{\mathbb{C}}$  are not linearly independent). Hence the  $\mathbb{R}$ -linear map  $\Phi : \mathbb{A} \rightarrow \mathbb{C}^k$  is open and therefore  $M_0 = M \cap \Phi^{-1}(\mathbb{C}_-^k)$  is dense in  $M$  since  $\mathbb{C}^k \setminus \mathbb{C}_-^k$  is nowhere dense in  $\mathbb{C}^k$ . Consider the holomorphic branch  $\psi : \mathbb{C}_- \rightarrow \mathbb{C}$  of the  $\sqrt{z}$  function satisfying  $\psi(1) = 1$  and let  $a \in M_0$ . Then  $\sigma(a) \subset \mathbb{C}_-$ . By the usual holomorphic functional calculus argument, we can take  $b = \psi(a) \in \mathbb{A}_{\mathbb{C}}$ . Then  $b^2 = a$  and since  $\psi$  is real on  $(0, \infty)$ , we have  $b = \psi(a) = \psi(a^\dagger) = b^\dagger$  and therefore  $b \in \mathbb{A}$ . Since  $a$  is invertible,  $b \in \mathbb{A}^*$  and  $a = S(b)$ . Thus  $M_0 \subseteq S(\mathbb{A}^*)$ . Since the homomorphism  $S : \mathbb{A}^* \rightarrow M$  is a local homeomorphism and  $\ker S = G_0$  is finite, the map  $S : \mathbb{A}^* \rightarrow M$  is open and closed. In particular,  $S(\mathbb{A}^*)$  is a closed subset of  $M$  containing the dense subset  $M_0$ . Thus  $S(\mathbb{A}^*) = \exp_{\mathbb{A}}(\mathbb{A}) = M$ .  $\square$

## 4 Proof of Theorems 1.2 and 1.3

It is straightforward to see that if  $G$  is a Hausdorff topological group and  $N$  is its discrete (=each point is isolated in  $G$ ) normal subgroup of  $G$ , then a subgroup  $K$  of  $G$  containing  $N$  is dense (respectively, nowhere dense) in  $G$  if and only if  $K/N$  is dense (respectively, nowhere dense) in  $G/N$ . Since a homomorphism  $\varphi : G \rightarrow G_0$  between Hausdorff topological groups  $G$  and  $G_0$  is a local homeomorphism if and only if  $\ker \varphi$  is discrete in  $G$  and the homomorphism  $\widehat{\varphi} : G/\ker \varphi \rightarrow G_0$ ,  $\widehat{\varphi}(g\ker \varphi) = \varphi(g)$  is a homeomorphism, we arrive to the following fact.

**Lemma 4.1.** *Let  $G$  and  $H$  be topological groups,  $\varphi : G \rightarrow H$  be a surjective homomorphism, which is also a local homeomorphism and  $K$  be a subgroup of  $G$  containing  $\ker \varphi$ . Then  $K$  is dense in  $G$  if and only if  $\varphi(K)$  is dense in  $H$ . Furthermore,  $K$  is nowhere dense in  $G$  if and only if  $\varphi(K)$  is nowhere dense in  $H$ .*

*Proof of Theorem 1.2.* By Lemma 3.2, the kernel  $K$  of the homomorphism  $\exp_{\mathbb{A}} : \mathbb{A} \rightarrow \mathbb{A}^*$  is a subgroup of  $(\mathbb{A}, +)$  generated by  $k = \kappa(\mathbb{A})$  linearly independent elements  $B_1, \dots, B_k$ . By Lemma 1.1,  $\dim_{\mathbb{R}} \mathbb{A} = 2n$ . Pick  $B_{k+1}, \dots, B_{2n} \in \mathbb{A}$  in such a way that  $B_1, \dots, B_{2n}$  is a basis in  $\mathbb{A}$  over  $\mathbb{R}$ . By Corollary 2.4, there is  $B_0 \in \mathbb{A}$  such that the additive semigroup  $H$  generated by  $B_0, \dots, B_{2n}$  is dense in  $\mathbb{A}$ . Since  $\exp_{\mathbb{A}} : \mathbb{A} \rightarrow \mathbb{A}$  is continuous and has dense in  $\mathbb{A}$  range  $\mathbb{A}^*$ ,  $\exp_{\mathbb{A}}(H)$  is dense in  $\mathbb{A}$ . Since  $e_j^B = I$  for  $1 \leq j \leq k$ , we see that  $\exp_{\mathbb{A}}(H) = \exp_{\mathbb{A}}(G)$ , where  $G$  is the subsemigroup of  $(\mathbb{A}, +)$  generated by  $\{B_0, B_{k+1}, B_{k+2}, \dots, B_{2n}\}$ . Hence  $\exp_{\mathbb{A}}(G)$  is dense in  $\mathbb{A}$ . Now let  $x$  be a cyclic vector for  $\mathbb{A}$ . Then  $\mathbb{A}x = \{Ax : A \in \mathbb{A}\} = \mathbb{C}^n$ . It follows that  $\exp_{\mathbb{A}}(G)x$  is dense in  $\mathbb{C}^n$ . That is,  $x$  is a hypercyclic vector for the  $m$ -tuple  $\{e^{B_0}, e^{B_{k+1}}, \dots, e^{B_{2n}}\}$  with  $m = 2n - k + 1$  of operators from  $\mathbb{A}$ . Thus  $\mathbb{A}$  contains a hypercyclic  $m$ -tuple.

Now let  $r$  be the minimal positive integer such that  $\mathbb{A}$  contains a hypercyclic  $r$ -tuple. It remains to show that  $r \geq m$ . Let  $\mathbf{T} = \{T_1, \dots, T_r\} \subset \mathbb{A}$  be a hypercyclic tuple and  $x \in \mathbb{C}^n$  be its hypercyclic vector. First, observe that each  $T_j$  must be invertible. Indeed, if  $1 \leq j \leq r$  and  $T_j$  is non-invertible, then  $L = T_j(\mathbb{C}^n) \neq \mathbb{C}^n$ . Then for each  $q \in \mathbb{Z}_+^r$  with  $q_j \neq 0$ ,  $T_1^{q_1} \dots T_r^{q_r} x \in L$ . Since  $L$ , being a proper linear subspace of  $\mathbb{C}^n$ , is nowhere dense and  $x$  is a hypercyclic vector for  $\mathbf{T}$ ,  $\{T_1^{q_1} \dots T_r^{q_r} x : q \in \mathbb{Z}_+^r, q_j = 0\}$

is dense in  $\mathbb{C}^n$ . Hence  $\mathbf{T} \setminus \{T_j\}$  is a hypercyclic  $(r-1)$ -tuple, which contradicts the minimality of  $r$ . Thus each  $T_j$  is invertible. Since  $\exp_{\mathbb{A}} : \mathbb{A} \rightarrow \mathbb{A}^*$  is onto, we can find  $A_1, \dots, A_r \in \mathbb{A}$  such that  $e^{A_j} = T_j$  for  $1 \leq j \leq r$ . Now let  $M$  be the subgroup of  $(\mathbb{A}, +)$  generated by  $A_1, \dots, A_r, B_1, \dots, B_k$ . Clearly,  $\exp_{\mathbb{A}}(M)x$  contains the  $\mathbf{T}$ -orbit of  $x$  and therefore is dense in  $\mathbb{C}^n$ . Since  $A \mapsto Ax$  is an onto linear map from  $\mathbb{A}$  to  $\mathbb{C}^n$  and  $\dim \mathbb{A} = n$ , it is a linear isomorphism. Hence the density of  $\exp_{\mathbb{A}}(M)x$  in  $\mathbb{C}^n$  implies the density of  $\exp_{\mathbb{A}}(M)$  in  $\mathbb{A}$  and therefore in  $\mathbb{A}^*$ . Since the additive subgroup  $M$  of  $\mathbb{A}$  contains the kernel  $K$  of  $\exp_{\mathbb{A}}$  and  $\exp_{\mathbb{A}} : \mathbb{A} \rightarrow \mathbb{A}^*$  is a surjective homomorphism and a local homeomorphism, Lemma 4.1 provides the density of  $M$  in  $\mathbb{A}$ . Since  $\dim_{\mathbb{R}} \mathbb{A} = 2n$  and  $M$  has  $r+k$  generators, Lemma 2.1 implies that  $r+k \geq 2n+1$  and therefore  $r \geq 2n-k+1 = m$ .  $\square$

*Proof of Theorem 1.3.* By Lemma 3.4, the topological group  $\mathbb{A}^*$  is the internal algebraic and topological direct product of its subgroups  $\exp_{\mathbb{A}}(\mathbb{A})$  and  $G$ , where  $G$  is isomorphic to  $\mathbb{Z}_2^m$  with  $m = \kappa_1(\mathbb{A})$ . Moreover, the kernel  $K$  of the homomorphism  $\exp_{\mathbb{A}} : \mathbb{A} \rightarrow \mathbb{A}^*$  is a subgroup of  $(\mathbb{A}, +)$  generated by  $k = \kappa_0(\mathbb{A})$  linearly independent elements  $B_1, \dots, B_k$ . By Lemma 1.1,  $\dim_{\mathbb{R}} \mathbb{A} = n$ . Pick  $B_{k+1}, \dots, B_n \in \mathbb{A}$  in such a way that  $B_1, \dots, B_n$  is a basis in  $\mathbb{A}$ . According to (1.2),  $2k+m \leq n$  and therefore  $m \leq n-2k \leq n-k$ . Since  $\mathbb{Z}_2^m$  has an  $m$ -element generating set and  $G$  is isomorphic to  $\mathbb{Z}_2^m$ , we can pick  $C_{k+1}, \dots, C_n \in G$  such that  $\{C_{k+1}, \dots, C_n\}$  is a generating subset of  $G$ . We also set  $C_j = I$  for  $0 \leq j \leq k$ . By Corollary 2.4, there is  $B_0 \in \mathbb{A}$  such that the subsemigroup  $N$  of  $G \times \mathbb{A}$  generated by  $(C_0, B_0), \dots, (C_n, B_n)$  is dense in  $G \times \mathbb{A}$ . Since  $\mathbb{A}^*$  is the internal algebraic and topological direct product of its subgroups  $\exp_{\mathbb{A}}(\mathbb{A})$  and  $G$  and  $\exp_{\mathbb{A}}$  is a local homeomorphism, the homomorphism  $\Phi : G \times \mathbb{A} \rightarrow \mathbb{A}^*$ ,  $\Phi(C, A) = Ce^A$  is surjective and is a local homeomorphism. Moreover,  $\ker \Phi = \{I\} \times K$ . Then  $\Phi(N)$  is dense in  $\mathbb{A}^*$  and therefore is dense in  $\mathbb{A}$ . Since  $C_j = I$  and  $e^{B_j} = I$  for  $1 \leq j \leq k$ ,  $\Phi(N)$  is precisely  $\mathcal{F}_{\mathbf{T}}$  for  $\mathbf{T} = (C_0 e^{B_0}, C_{k+1} e^{B_{k+1}}, \dots, C_n e^{B_n})$ . The density of  $\mathcal{F}_{\mathbf{T}}$  in  $\mathbb{A}$  implies that each cyclic vector  $x$  for  $\mathbb{A}$  is also a hypercyclic vector for  $\mathbf{T}$ , which happens to be an  $r$ -tuple of elements of  $\mathbb{A}$  with  $r = n-k+1$ .

Now let  $p$  be the minimal positive integer such that  $\mathbb{A}$  contains a  $p$ -tuple with an orbit, which is not nowhere dense. It remains to show that  $p \geq r$ . Let  $\mathbf{T} = \{T_1, \dots, T_p\} \subset \mathbb{A}$  for which there is  $x \in \mathbb{R}^n$ , whose orbit  $O(\mathbf{T}, x)$  is not nowhere dense. Exactly as in the proof of Theorem 1.3, minimality of  $p$  implies that each  $T_j$  is invertible. Let  $\mathbf{T}^{[2]} = \{T_1^2, \dots, T_p^2\}$ . Then  $O(\mathbf{T}, x)$  is the union of  $O(\mathbf{T}^{[2]}, x_{\varepsilon})$ , where  $\varepsilon \in \{0, 1\}^p$  and  $x_{\varepsilon} = T_1^{\varepsilon_1} \dots T_p^{\varepsilon_p} x$ . Thus there is  $y \in \{x_{\varepsilon} : \varepsilon \in \{0, 1\}^p\}$  such that  $O(\mathbf{T}^{[2]}, y)$  is not nowhere dense. Then  $y$  is a cyclic vector for  $\mathbb{A}$  and therefore the map  $T \mapsto Ty$  from  $\mathbb{A}$  to  $\mathbb{R}^n$  is onto. Since  $\dim \mathbb{A} = n$ , this map is a linear isomorphism. Hence the semigroup  $\mathcal{F}_{\mathbf{T}^{[2]}}$  is not nowhere dense in  $\mathbb{A}$ . Thus the subgroup  $\mathcal{F}_{\mathbf{T}^{[2]}}^0$  of  $\mathbb{A}^*$  generated by  $\mathbf{T}^{[2]}$  is also not nowhere dense in  $\mathbb{A}$ . By Lemma 3.3, there are  $A_1, \dots, A_p \in \mathbb{A}$  such that  $T_j^2 = e^{A_j}$  for  $1 \leq j \leq p$ . Let  $Q$  be the additive subgroup of  $\mathbb{A}$  generated by  $A_1, \dots, A_p, B_1, \dots, B_k$ . Since  $e^{A_j} = T_j^2$  for  $1 \leq j \leq p$  and  $e^{B_j} = I$  for  $1 \leq j \leq k$ ,  $\exp_{\mathbb{A}}(Q) = \mathcal{F}_{\mathbf{T}^{[2]}}^0$  is a subgroup of  $\mathbb{A}^*$ , which is not nowhere dense. Since  $\exp_{\mathbb{A}} : \mathbb{A} \rightarrow \exp_{\mathbb{A}}(\mathbb{A})$  is a surjective homomorphism and a local homeomorphism and  $Q$  contains  $\ker \exp_{\mathbb{A}}$ , Lemma 4.1 implies that  $Q$  is not nowhere dense in  $\mathbb{A}$ . Since  $Q$  is generated by  $p+k$  elements and  $\dim \mathbb{A} = n$ , Lemma 2.1 implies that  $p+k \geq n+1$  and therefore  $p \geq n-k+1 = r$ .  $\square$

## 5 Proof of Propositions 1.6 and 1.7

*Proof of Proposition 1.6.* Consider the 3-dimensional linear subspace  $\mathbb{A}$  of  $L(\mathbb{K}^3)$  defined as follows:

$$\mathbb{A} = \{A_z : z \in \mathbb{K}^3\}, \quad \text{where } A_z = \begin{pmatrix} z_1 & 0 & 0 \\ z_2 & z_1 & 0 \\ z_3 & 0 & z_1 \end{pmatrix}$$

It is easy to see that  $\mathbb{A}$  is a commutative subalgebra of  $L(\mathbb{K}^3)$  and  $\mathbb{A}$  is cyclic with the cyclic vector  $(1, 0, 0)$ . Moreover, there is exactly one character on  $\mathbb{A}$ :  $A_z \mapsto z_1$ . Hence  $\kappa(\mathbb{A}) = 1$  if  $\mathbb{K} = \mathbb{C}$  and  $\kappa_0(\mathbb{A}) = 0$ ,  $\kappa_1(\mathbb{A}) = 1$  if  $\mathbb{K} = \mathbb{R}$ . According to Theorems 1.2 and 1.3,  $\mathbb{A}$  contains a hypercyclic



6-tuple (respectively, 4-tuple)  $\mathbf{T}$  if  $\mathbb{K} = \mathbb{C}$  (respectively,  $\mathbb{K} = \mathbb{R}$ ). Since any hypercyclic tuple of operators on  $\mathbb{K}^3$  is contained in precisely one commutative cyclic subalgebra of  $L(\mathbb{K}^3)$ , which is the unital subalgebra generated by the tuple,  $\mathbb{A}_{\mathbf{T}} = \mathbb{A}$ . By Lemma 1.1, any operator commuting with  $\mathbf{T}$  belongs to  $\mathbb{A}$ . Thus it suffices to show that  $\mathbb{A}$  contains no cyclic operators. Indeed, assume that there is  $z \in \mathbb{K}^3$  such that  $A_z$  is cyclic. Cyclicity of  $A_z$  implies cyclicity of  $A_z - z_1 I$ . Since the last two columns of  $A_z - z_1 I$  are zero, the rank of  $A_z - z_1 I$  is at most 1. Since the rank of every cyclic operator on  $\mathbb{C}^n$  is at least  $n - 1$ , we have arrived to a contradiction.  $\square$

*Proof of Proposition 1.7.* Let  $a_1, a_2, a_3 > 0$ ,  $b_1, b_2, b_3 \in \mathbb{R}$  and  $T_j = \begin{pmatrix} a_j & b_j \\ 0 & a_j \end{pmatrix} \in L(\mathbb{R}^2)$  for  $1 \leq j \leq 3$ .

It is straightforward to verify that for every  $x = (x_1, x_2) \in \mathbb{R}^2$  and each  $n_1, n_2, n_3 \in \mathbb{Z}_+$ ,

$$T_1^{n_1} T_2^{n_2} T_3^{n_3} x = a_1^{n_1} a_2^{n_2} a_3^{n_3} \left( x_1 + x_2 \left( \frac{b_1 n_1}{a_1} + \frac{b_2 n_2}{a_2} + \frac{b_3 n_3}{a_3} \right), x_2 \right).$$

Thus for  $\mathbf{T} = (T_1, T_2, T_3)$ , the  $\mathbf{T}$ -orbit  $O(\mathbf{T}, x)$  of  $x$  is contained in the half-plane  $\Pi = \{(s, t) \in \mathbb{R}^2 : t > 0\}$  provided  $x_2 > 0$ . It remains to show that  $a_j > 0$  and  $b_j \in \mathbb{R} \setminus \{0\}$  can be chosen in such a way that  $O(\mathbf{T}, x)$  is dense in  $\Pi$  if  $x_2 > 0$ .

According to the above display, the density of  $O$  in  $\Pi$  is equivalent to the density of

$$N = \left\{ \left( \frac{b_1 n_1}{a_1} + \frac{b_2 n_2}{a_2} + \frac{b_3 n_3}{a_3}, a_1^{n_1} a_2^{n_2} a_3^{n_3} \right) : n_1, n_2, n_3 \in \mathbb{Z}_+ \right\}$$

in  $\Pi$ . Since the map  $(t, s) \mapsto (t, \ln s)$  is a homeomorphism from  $\Pi$  onto  $\mathbb{R}^2$ , the density of  $O$  in  $\Pi$  is equivalent to the density of

$$N_0 = \left\{ n_1 \left( \frac{b_1}{a_1}, \ln a_1 \right) + n_2 \left( \frac{b_2}{a_2}, \ln a_2 \right) + n_3 \left( \frac{b_3}{a_3}, \ln a_3 \right) : n_1, n_2, n_3 \in \mathbb{Z}_+ \right\}$$

in  $\mathbb{R}^2$ . Now choose any  $a_1, a_2 > 0$  and  $b_1, b_2 \in \mathbb{R} \setminus \{0\}$  such that the vectors  $(\frac{b_1}{a_1}, \ln a_1)$  and  $(\frac{b_2}{a_2}, \ln a_2)$  are linearly independent. By Corollary 2.4, there are  $a_3 > 0$  and  $b_3 \in \mathbb{R} \setminus \{0\}$  such that the additive subsemigroup  $N_0$  in  $\mathbb{R}^2$  generated by  $(\frac{b_1}{a_1}, \ln a_1)$ ,  $(\frac{b_2}{a_2}, \ln a_2)$  and  $(\frac{b_3}{a_3}, \ln a_3)$  is dense in  $\mathbb{R}^2$ . For this choice of  $a_j$  and  $b_j$ , the orbit  $O(\mathbf{T}, x)$  is dense in  $\Pi$  whenever  $x_2 > 0$ .  $\square$

## 6 Remarks

It is easy to extend Proposition 1.6 to provide hypercyclic tuples  $\mathbf{T}$  on  $\mathbb{K}^n$  with  $n \geq 3$  such that there are no cyclic operators commuting with  $\mathbf{T}$ . One can also produce an infinite dimensional version. Namely, let  $X$  be any infinite dimensional separable Fréchet space over  $\mathbb{K}$  and  $Y$  be a closed linear subspace of  $X$  of codimension 3. Then  $X$  can be naturally interpreted as  $Y \oplus \mathbb{K}^3$ . Take the tuple  $\mathbf{T}$  of operators on  $\mathbb{K}^3$  constructed in the proof of Proposition 1.6. Since every separable infinite dimensional separable Fréchet space possesses a mixing operator [1], we can pick a mixing  $S \in L(Y)$ . It is now easy to verify that the tuple  $\mathbf{T}_0 = \{S \oplus T : T \in \mathbf{T}\}$  is a hypercyclic tuple of operators on  $Y \oplus \mathbb{K}^3 = X$  consisting of the same number of operators as  $\mathbf{T}$ . Take any  $T \in \mathbf{T}$ . Then there is a unique  $\lambda \in \mathbb{K}$  such that  $(T - \lambda I_{\mathbb{K}^3})^3 = 0$ . Since  $(S - \lambda I_Y)^3(Y)$  is dense in  $Y$ , we have  $Y = \overline{(T_0 - \lambda I_X)^3(X)}$ , where  $T_0 = S \oplus T$ . Since  $T_0 \in \mathbf{T}_0$ , the equality  $Y = \overline{(T_0 - \lambda I_X)^3(X)}$  implies that  $Y$  is invariant for every operator in  $L(X)$  commuting with  $\mathbf{T}_0$ . By factoring  $Y$  out, we see that the existence of a cyclic operator on  $X$  commuting with  $\mathbf{T}_0$  implies the existence of a cyclic operator on  $\mathbb{K}^3$  commuting with  $\mathbf{T}$ . Since the latter does not exist, we arrive to the following result.

**Proposition 6.1.** *Every infinite dimensional separable complex (respectively, real) Fréchet space admits a hypercyclic 6-tuple (respectively, a 4-tuple)  $\mathbf{T}$  of operators such that there are no cyclic operators commuting with  $\mathbf{T}$ .*

Summarizing, we notice that any separable Fréchet space  $X$  of dimension  $\geq 3$  possesses a hypercyclic tuple  $\mathbf{T}$  of operators such that there are no cyclic operators commuting with  $\mathbf{T}$ . Since any operator on a one-dimensional space is cyclic, this does not carry on to the case  $\dim X = 1$ . Neither

it does to the case  $\dim X = 2$ . Indeed, the only non-cyclic operators on  $\mathbb{K}^2$  have the form  $\lambda I$  with  $\lambda \in \mathbb{K}$ . Since a hypercyclic tuple of operators on  $\mathbb{K}^2$  can not consist only of such operators, we arrive to the following proposition.

**Proposition 6.2.** *Every hypercyclic tuple of operators on  $\mathbb{K}^2$  contains a cyclic operator.*

Theorem 5.5 in [3] provides a sufficient condition on a commuting tuple  $\mathbf{T}$  of continuous linear operators on a locally convex topological vector space  $X$  to have each orbit either dense or nowhere dense. It is worth noting that the local convexity condition can be dropped in exactly the same way as it is done by Wengenroth [4]. One just has to replace the point spectrum  $\sigma_p(A^*)$  of the dual operator  $A^*$  of each  $A \in L(X)$  in the statement by the set  $\{\lambda \in \mathbb{K} : \overline{(A - \lambda I)(X)} \neq X\}$ , which coincides with  $\sigma_p(A^*)$  in the locally convex case.

Finally, we note that Feldman [3] proved that there are no hypercyclic tuples of normal operators on a separable infinite dimensional complex Hilbert space  $\mathcal{H}$ . On the other hand, Bayart and Matheron [2] constructed a unitary operator  $U$  on  $\mathcal{H}$  and  $x \in \mathcal{H}$  such that  $\{zU^n x : z \in \mathbb{C}, n \in \mathbb{Z}_+\}$  is dense in  $\mathcal{H}$  endowed with its weak topology  $\sigma$ . According to Feldman [3], there are  $a, b \in \mathbb{C}$  such that  $\{a^n b^m : n, m \in \mathbb{Z}_+\}$  is dense in  $\mathbb{C}$ . It follows that  $(aI, bI, U)$  is a commuting triple of normal operators on  $\mathcal{H}$ , which is a hypercyclic triple of operators on  $(\mathcal{H}, \sigma)$ .

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